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Analysis of Optimal Turning Maneuvers in the Vertical Plane

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Maneuverability in the vertical plane is of great importance in the performance assessment of a high-thrust fighter aircraft. In this paper, two basic vertical turning maneuvers, frequently used in air combat, are analyzed. Optimality of these maneuvers is defined as "minimum time" combined with specific-energy considerations. The exact solution of such two-point boundary-value problems (TPBVP) requires iterative computation which is prohibitive for a "real-time" airborne application and an excessive effort for parametric analysis. The paper concentrates on topics which provide an insight into vertical maneuvering in air combat, but need only minimal computational effort: 1) qualitative investigation of optimal subarcs (including optimal thrust-switching sequence); 2) analysis of standard maneuver strategies; and 3) suboptimal "feedback"-type solutions yielding a fine approximation for the exact one of the original TPBVP.

Nomenclature

a = velocity of sound
C_D = drag coefficient
C_{D_0} = zero-lift drag coefficient
C_L = lift coefficient
D = aerodynamic drag force
D_0 = zero-lift drag force
D_i = induced drag for level flight
E = specific energy
g = acceleration of gravity
h = altitude
H = variational Hamiltonian
J = payoff function
k_E = specific-energy weighting coefficient
K = induced drag parameter
L = aerodynamic lift force
M = Mach number
n = aerodynamic load factor
n_L = lift-limited load factor, Eq. (16)
$n_{\max}$ = structural load factor limit
q = dynamic pressure
S = aircraft reference wing surface
t = time
T = thrust
V = aircraft true airspeed
W = aircraft weight
x = horizontal distance
γ = flight path angle in the vertical plane
ϵ = small parameter
η = throttle parameter
$\lambda_E, \lambda_\gamma, \lambda_h, \lambda_x$ = components of the costate vector
μ_L, μ_n, μ_T = constraint multipliers
ρ = air density
τ = stretched time scale

Subscripts
f = final value
0 = initial value
c = critical value
max = maximum value
SW = switch value

Superscripts
$(\ )^i$ = inner solution
$(\ )^o$ = outer solution
$(\ )^*$ = optimal value

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Introduction

THE development of a new generation of fighter airplanes in the last decade has motivated the analysis of optimal aircraft maneuvers. The main research activity has been focused on the investigation of optimal climb peformance,[1-3] and "energy turns"[4,5] (which can be considered as almost horizontal maneuvers). Although vertical turning maneuvers, such as "loop," "Immelman" and "split-S" (shown in Figs. 1 and 2), have been standard aerobatic exercises and an inherent part of fighter pilot training, their analysis has not been emphasized. The first published works dealing directly with this topic[6,7] date from 1978. The paper of Uehara et al.,[7] based on the Ph.D. dissertation of the first author,[8] deals with the minimum-time full "loop" (see Fig. 1a) and is of considerable theoretical interest.

Vertical maneuverability recently became a factor of increasing importance for modern fighter aircraft with high thrust-to-weight ratios (F-15, F-16, etc.). Air combat experience indicates that such airplanes have a definite tendency to use vertical maneuver planes. Analysis of vertical turning maneuvers can therefore be a useful tool in evaluating and comparing modern fighters. These maneuvers seem to be more representative for air combat evaluation than the traditionally used horizontal turning performance.

The present paper deals with the analysis of two basic vertical turning maneuvers frequently used in air-to-air engagements, the "half loop" and the "split-S" (see Fig. 2). These two maneuvers can be considered as the decomposition of a "full loop," a well-known aerobatic exercise, analyzed by Uehara et al.[7,8] They have different roles in air combat and therefore are treated separately. Both maneuvers consist of a 180-deg of change in flight path angle, which also creates a 180-deg change in azimuth. The "half loop" is initiated as a pull-up maneuver from horizontal flight. The split-S is a pull-down maneuver from an inverted horizontal initial condition. Both maneuvers are described by the same set of differential equations but with different end condition. The split-S can

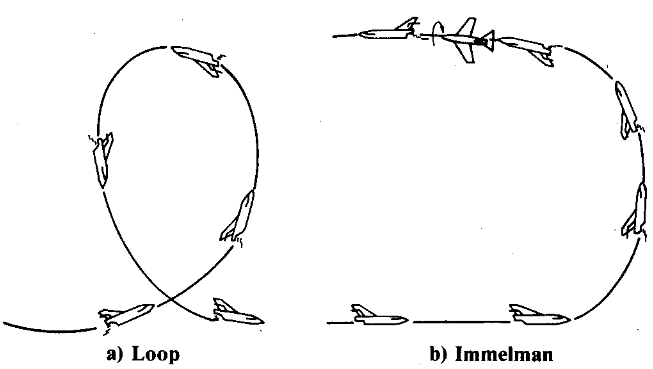

Fig. 1 Vertical turning maneuvers.

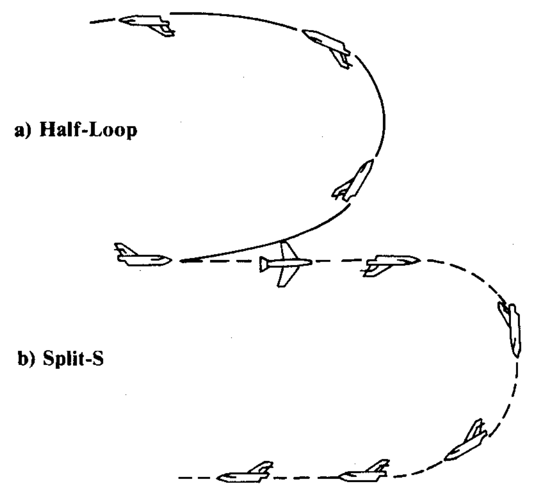

Fig. 2 "Half loop" and "Split-S."

also be considered as the inverted mirror image of the half-loop.

Along the pull-up trajectory of the half loop, velocity is always decreasing and care has to be taken to reach the top with sufficient airspeed for airplane controllability. In a split-S the aircraft is generally accelerating, but excessive speed may reduce turning capability or even violate flight envelope limits.

In air combat, vertical turning maneuvers are used both in defensive and offensive roles. In most cases it is of major importance to complete the turn in minimum time. Mimimum-time turns, however, require "hard" maneuvering, creating high induced drag and, as a consequence, lead to important losses of specific energy.

Fighter pilots tend to avoid excessive specific-energy losses in air combat. This fact has to be taken into account in the determination of the appropriate payoff functions for an optimal vertical-turning maneuver. There are two alternatives to express in a quantitative form the pilot requirement for rapid maneuvering with qualitative consideration for specific-energy saving: 1) formulate the optimal vertical turning maneuver as a mimimum-time problem with a prescribed final specific energy; 2) define the payoff function to be minimized as the weighted sum of the final time and specific energy change. Both formulations require solution of a nonlinear optimal control problem of at least three state variables, which is unfortunately a two-point boundary-value problem. The solution can be obtained in an open-loop form by several iterative numerical techniques, requiring, even if an efficient algorithm[9] is used, an extensive amount of computation for each set of initial and terminal conditions. As a consequence, this approach has a very limited practical value either for systematic performance assessment, where a large number of computations has to be carried out varying aircraft design parameters and initial conditions, or for real-time airborne computation, where a closed-loop (feedback) optimal control law is required. For such purposes, approximate (suboptimal) solutions might be preferred.

The objective of this paper is to generate an insight into the relatively complex problem of vertical maneuvering. Attention is focused on topics that do not require significant numerical effort. First, the structure of the optimal solution is derived and investigated qualitatively, giving special attention to the conditions of optimal thrust switching. Analysis is based on a representative, realistic mathematical model. Next, the usefulness of three simple "standard" maneuver strategies, proposed for comparing vertical turning maneuvers, are discussed. In a following section, two approximate "feedback"-type solutions of the optimal control problem, detailed in separate reports,[6,10] are briefly summarized. Suboptimal results are compared to the exact solution of the original two-point boundary-value problem computed by a recently developed combination of numerical algorithms.[9]

Equations of Motion

The set of differential equations describing the motion of a point-mass vehicle in the vertical plane over a flat, nonrotating earth, assuming small angles of attack, is

$$\dot{E}=V(T-D)/W \tag{1}$$

$$\dot{\gamma}=(g/V)(L/W-\cos\gamma) \tag{2}$$

$$\dot{h}=V\sin\gamma \tag{3}$$

$$\dot{x}=V\cos\gamma \tag{4}$$

The specific energy is defined as

$$E=h+V^2/2g \tag{5}$$

and the other variables are shown in Fig. 3. The aerodynamic and propulsive forces are determined by the following equations:

$$T=\eta T_{\max}(h,V) \tag{6}$$

$$L=\tfrac{1}{2}\rho(h)V^2 S C_L \tag{7}$$

$$D=\tfrac{1}{2}\rho(h)V^2 S C_D(M,C_L) \tag{8}$$

For parabolic drag polar

$$C_D(M,C_L)=C_{D_0}(M)+K(M)C_L^2 \tag{9}$$

and using the definition of the aerodynamic load factor,

$$n\triangleq\frac{L}{W}=\frac{\tfrac{1}{2}\rho(h)V^2C_L}{(W/S)} \tag{10}$$

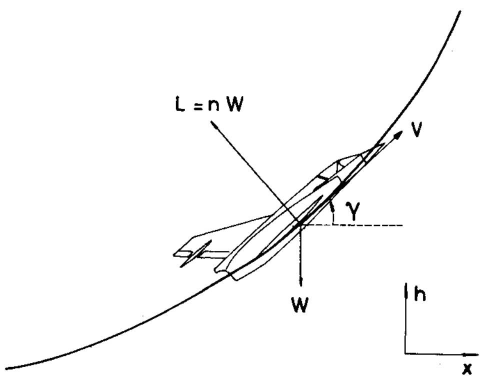

Fig. 3 Variables in vertical motion.

the drag force can be expressed as

$$D = D_0 + n^2 D_i \tag{11}$$

where D_0 is the zero-lift drag and D_i stands for the induced drag in level flight

$$D_i = \frac{KW^2}{\frac{1}{2}\rho(h) V^2 S} \tag{12}$$

The fact that the maneuver takes place in the airplane "dynamic-flight envelope" requires that the following state contraints be satisfied:

Minimum altitude limit

$$h > 0 \tag{13}$$

Maximum dynamic pressure limit

$$q = \frac{1}{2}\rho(h) V^2 \leq q_{\max} \tag{14}$$

Maximum Mach number limit

$$V \leq a(h) M_{\max} \tag{15}$$

Loft ceiling[11] limit, expressed by

$$n(C_{L_{\max}}) \stackrel{\Delta}{=} n_L \geq 1 \tag{16}$$

A point-mass airplane in the vertical plane is governed by two independent control variables: the throttle parameter η and the lift coefficient C_L or alternatively, the aerodynamic-load factor n, which are subject to the following constraints:

$$0 \leq \eta \leq 1 \tag{17}$$

$$|C_L| \leq C_{L_{\max}}(M) \tag{18}$$

$$|n| \leq n_{\max} \tag{19}$$

If the load factor is used as control variable, Eq. (18) has to be written as a state-variable-dependent constraint

$$|n| \leq n_L(V,h) = \frac{\frac{1}{2}\rho V^2 C_{L_{\max}}(M)}{(W/S)} \tag{20}$$

or by substituting Eq. (5),

$$|n| \leq n_L(E,h) = g\rho(h)[E-h]\frac{C_{L_{\max}}(h,E)}{W/S} \tag{21}$$

This mathematical model is different from the one used in the first part of Ref. 7, where constant atmospheric density and unlimited load factor are assumed.

For the maneuvers analyzed in this paper, the initial flight-path angle is always zero:

$$\gamma_0 = 0 \tag{22}$$

The final value in the case of a "half-loop" is

$$\gamma_f = \pi \tag{23}$$

For a split-S, the inverted mirror-image formulation (see Fig. 2) yields

$$\gamma_f = -\pi \tag{24}$$

The terminal coordinates of the trajectory (h_f, x_f) generally are not specified.

Formal Optimal Control Solution

The optimal control problem for the vertical turning maneuvers to be analyzed is formulated as follows:

> Given the dynamic system described by Eqs. (1-4), with specified initial conditions (E_0, x_0, h_0), *and Eq. (22), determine the control functions* $\eta^*(t)$ and $n^*(t)$, subject to the constraints in Eqs. (17-19), which transfer the system—without violating the state constraints in Eqs. (13-16)—to a final state $(E_f, \gamma_f, x_f, h_f)$, minimizing one of the alternative payoff functions:
>
> $$J_1 = t_f \tag{25}$$
>
> with prescribed final specific energy E_f.
>
> $$J_2 = t_f + k_E(E_0 - E_f) \tag{26}$$
>
> with unspecified energy at t_f.
> The final value of γ is given either by Eq. (23) or by Eq. (24); h_f and x_f are not specified.

Assuming that the state constraints are not violated, the variational Hamiltonian is written as

$$H = -1 + \lambda_E V\left(\frac{\eta T_{\max} - D_0 - n^2 D_i}{W}\right) + \lambda_\gamma \frac{g}{V}(n\text{-}\cos\gamma)$$
$$+ \lambda_h V\sin\gamma + \lambda_x V\cos\gamma + \mu_T(\eta - \eta^2)$$
$$+ \mu_n(n_{\max}^2 - n^2) + \mu_L(n_L^2 - n^2) \tag{27}$$

The necessary conditions for optimality[12] require that the costate variables satisfy the adjoint set of differential equations

$$\begin{aligned} \dot\lambda_E &= -\partial H/\partial E|_h \\ \dot\lambda_\gamma &= -\partial H/\partial\gamma \\ \dot\lambda_h &= -\partial H/\partial h|_E \\ \dot\lambda_x &= -\partial H/\partial x = 0 \end{aligned} \tag{28}$$

with the boundary conditions

$$\lambda_h(t_f) = \lambda_x(t_f) = 0 \tag{29}$$

The value of $\lambda_\gamma(t_f)$ cannot be determined a priori since γ_f is specified. The same is true for $\lambda_E(t_f)$ if E_f is prescribed, as for J_1 in Eq. (25). If, however, Eq. (26) is used as the optimal payoff function, then

$$\lambda_E(t_f) = -\frac{\partial J_2}{\partial E_f} = k_E \tag{30}$$

As a consequence of Eqs. (28) and (29),

$$\lambda_x \equiv 0 \tag{31}$$

The other costate equations are written explicitly as

$$\dot\lambda_E = -\frac{g}{V}\frac{\partial H(V,h)}{\partial V}$$
$$= -\frac{\lambda_E g}{WV}\left[(\eta T_{\max} - D) + \left(\eta\frac{\partial T_{\max}}{\partial V} - \frac{\partial D}{\partial V}\right)V\right]$$
$$+ \lambda_\gamma \frac{g^2}{V^3}(n - \cos\gamma) - \lambda_h \frac{g}{V}\sin\gamma - 2\mu_L \frac{g}{V} n_L \frac{\partial n_L}{\partial V} \tag{32}$$

$$\dot{\lambda}_\gamma = -\lambda_\gamma \frac{g}{V}\sin\gamma - \lambda_h V\cos\gamma \tag{33}$$

$$\dot{\lambda}_h = -\frac{\partial H(V,h)}{\partial h} + \frac{g}{V}\frac{\partial H(V,h)}{\partial V}$$
$$= -\lambda_E \frac{V}{W}\left(\eta\frac{\partial T_{\max}}{\partial h} - \frac{\partial D}{\partial h}\right) - 2\mu_L n_L \frac{\partial n_L}{\partial h} - \dot{\lambda}_E \tag{34}$$

The optimal control functions $\eta^*(t)$ and $n^*(t)$ have to maximize the Hamiltonian Eq. (27):

$$\eta^*, n^* = \arg\max_{\eta,\,n} H(E,\gamma,h,\lambda_E,\lambda_\gamma,\lambda_h,\eta,n) \tag{35}$$

Formal differentiation yields

$$\frac{\partial H}{\partial \eta} = \lambda_E \frac{V}{W} T_{\max} + \mu_T(1-2\eta) = 0 \tag{36}$$

$$\frac{\partial H}{\partial n} = -2\lambda_E \frac{VD_i}{W} n + \lambda_\gamma \frac{g}{V} - 2n(\mu_n + \mu_L) = 0 \tag{37}$$

The control constraint multipliers have to satisfy

$$\mu_T = 0 \quad \text{if} \quad 0<\eta<1 \tag{38}$$

$$\mu_n = 0 \quad \text{if} \quad |n| < n_{\max} \tag{39}$$

$$\mu_L = 0 \quad \text{if} \quad |n| < n_L(V,h) \tag{40}$$

As a consequence, the optimal throttle parameter for $\lambda_E \neq 0$ is

$$\eta^* = \tfrac{1}{2}[\text{sign}\,\lambda_E + 1] \tag{41}$$

and

$$\mu_T = V(T_{\max}/W)\,|\lambda_E| \tag{42}$$

The optimal unconstrained load factor is obtained from Eq. (37) using Eq. (12),

$$n^* = n_I^* \overset{\Delta}{=} \frac{\lambda_\gamma}{2\lambda_E}\,\frac{g}{V^2}\,\frac{W}{D_i} = \frac{\lambda_\gamma}{\lambda_E}\,\frac{g\rho(h)}{4K(W/S)} \tag{43}$$

If

$$n_{\max} < |n_I^*| < n_L(V,h)$$
$$n^* = n_m^* \overset{\Delta}{=} n_{\max}\,\text{sign}[n_I^*] \tag{44}$$

and

$$\mu_n = \lambda_\gamma(g/2Vn_m^*) - \lambda_E(VD_i/W) \tag{45}$$

If

$$n_{\max} > |n_I^*| > n_L(V,h)$$
$$n^* = n_L^* \overset{\Delta}{=} n_L(V,h)\,\text{sign}[n_I^*] \tag{46}$$

and

$$\mu_L = \lambda_\gamma(g/2Vn_L^*) - \lambda_E(VD_i/W) \tag{47}$$

Since time does not appear explicitly in the equations and the final time t_f is not prescribed,

$$H^*(\eta^*, n^*) \equiv 0 \tag{48}$$

In order to ensure that the Hamiltonian is indeed maximized, the matrix of its second-order partial derivatives respective to the control variables has to be negative semidefinite. Since the cross derivatives are zero, this condition is reduced to

$$\frac{\partial^2 H}{\partial \eta^2} = -2\mu_T \leq 0 \tag{49}$$

$$\frac{\partial^2 H}{\partial n^2} = -2\left(\lambda_E V\frac{D_i}{W} + \mu_n + \mu_L\right) \leq 0 \tag{50}$$

In the view of Eqs. (38) and (42), Eq. (49) is always satisfied. In Eq. (50) strict inequality is guaranteed if $\lambda_E > 0$ and $n^* = n_I^*$. If the optimal load factor is one of the constraint boundaries, the necessary condition of Eq. (50) has the form of

$$\frac{\partial^2 H}{\partial n^2} = -\frac{\lambda_\gamma}{n^*}\,\frac{g}{V} \leq 0 \tag{51}$$

requiring that λ_γ and the optimal load factor have the same sign. For $\lambda_E \leq 0$, inspection of Eqs. (50) and (51) indicates that only constrained load factors are maximizing. If $\lambda_E = 0$, there is a possibility for an intermediate thrust subarc, investigated in the Appendix, and it is shown that such singular subarc is not optimal.

Optimal Thrust Switching Sequence

Nonoptimality of intermediate thrust leaves the optimal thrust-control parameter η^* to be determined by Eq. (41), i.e.,

$$\lambda_E > 0 \qquad \eta^* = 1$$
$$\lambda_E < 0 \qquad \eta^* = 0 \tag{52}$$

Thrust is switched whenever λ_E changes sign. The direction of the change is determined by the sign of $\dot{\lambda}_E$. Since the optimal trajectory has no singular thrust subarc, λ_E and $\dot{\lambda}_E$ cannot be zero simultaneously.

In view of Eqs. (37) and (43), $\lambda_E = 0$ requires either constrained load factor or $\lambda_\gamma = 0$. Simultaneous zero value for both costate variables is rather unlikely. If this happens, the Hamiltonian becomes independent of the load factor and its optimal value cannot be determined. It is easy to see that for all realistic maneuvers, λ_γ keeps a constant sign (positive for a loop, negative for the split-S). Thus, substituting $\lambda_E = 0$ into Eq. (32), two cases have to be investigated separately.

$n^* = n_m^*(\mu_L = 0)$

For this case, Eq. (32) yields at the moment of thrust switch

$$\dot{\lambda}_E(t_{SW}) = \lambda_\gamma(g^2/V^3)(n^* - \cos\gamma) - (g/V)\lambda_h \sin\gamma \tag{53}$$

It has been found from numerical analysis[9] that the term including λ_h is at least one order of magnitude smaller than the other one. It is not unexpected since the problem is essentially one of turning, and altitude change has only secondary importance. As a consequence, the sign of $\dot{\lambda}_E$ is determined by the first term of Eq. (53).

$$\dot{\lambda}_E(t_{SW}) \simeq \lambda_\gamma(g^2/V^3)(n^* - \cos\gamma) > 0 \tag{54}$$

indicating that

$$\lambda_E(t<t_{SW}) < 0 \rightarrow \eta(t<t_{SW}) = 0$$
$$\lambda_E(t>t_{SW}) > 0 \rightarrow \eta(t>t_{SW}) = 1 \tag{55}$$

Thus, in a structural load-factor-limited subarc, the only possible change in the thrust is from zero to $T_{\max}$.

$n^* = n_L^* (\mu_L \neq 0)$

In lift-limited maneuver subarc, substituting Eq. (47) yields

$$\dot{\lambda}_E(t_{SW}) = -(g^2/V^3)\lambda_\gamma(n_L^* + \cos\gamma) - (g/V)\lambda_h \sin\gamma \quad (56)$$

Again neglecting the second term, which is rather small, and taking into account Eq. (51), one obtains

$$\dot{\lambda}_E(t_{SW}) \simeq -(g^2/V^3)\lambda_\gamma(n_L + \cos\gamma) < 0 \quad (57)$$

showing that

$$\lambda_E(t<t_{SW}) > 0 \rightarrow \eta(t<t_{SW}) = 1$$

$$\lambda_E(t>t_{SW}) < 0 \rightarrow \eta(t>t_{SW}) = 0 \quad (58)$$

In such a subarc, thrust can be changed only by throttling to zero.

The different types of admissible optimal control combinations are summarized in Table 1.

The results presented in this section agree with Ref. 7—in spite of the differences in the formulation as well as in the mathematical model—in all points for which a relevant comparison can be made.

Analysis of "Standard" Maneuver Strategies

The necessary conditions of optimality provide enough equations to solve the optimal control problem stated previously. However, since some of the end conditions are given at $t=0$ and others at $t=t_f$, a two-point boundary-value problem (TPBVP) has to be solved. The solution of the TPBVP in the general case has to be obtained by iterative methods.

There exist many algorithms for numerical solution of such a problem, all of which require a large amount of iterative computation. Even if an efficient combination of several algorithms[9] is used, reducing the computational effort by more than 50%, the iterative approach is hardly practical and often prohibitive for systematical performance assessment of competing airplane designs. For this purpose, simple "standard" maneuver strategies have been used as a basis of comparison. Three different types of "standard" strategies are described and discussed briefly in this Section.

"Hard" Vertical Turns

If the terminal specific energy is of no particular importance, the original optimization problem is reduced to maximize, at any given state (E,γ,h), the vertical turning rate $\dot{\gamma}$ in Eq. (2), leading to maximization of the load factor n. The optimal value of the load factor is determined by the relevant one of the constraints in Eqs. (19) and (20). If the critical value of the dynamic pressure

$$q = \tfrac{1}{2}\rho(h)V^2 = g\rho(h)[E-h] \quad (59)$$

is defined as

$$q_c = \frac{W/S}{(C_L)_{\max}} n_{\max} \quad (60)$$

Table 1 Optimal control combinations

n^*	η^*	Thrust switch
n_1^*	1	none
n_m^*	0, 1	$0 \rightarrow 1$
n_L^*	0, 1	$1 \rightarrow 0$

the optimal load factor is given by

$$n^*(q<q_c) = n_L \operatorname{sign}(\dot{\gamma})$$

$$n^*(q \geq q_c) = n_{\max} \operatorname{sign}(\dot{\gamma}) \quad (61)$$

Sign$(\dot{\gamma})$ is positive for a pull-up maneuver and negative for a split-S.

This "hard" turning strategy is the solution of the original optimal control problem with $\lambda_E(t_f)=0$ (i.e., with free terminal specific energy or with $k_E=0$).

As n^* does not depend on the costate variables, the optimal trajectory can be found by integrating the state equations (1-4), assuming full or zero thrust. This assumption has to be verified by integrating the costate equations (32-34) backwards, with the boundary conditions $\lambda_{E_f} = \lambda_{h_f} = \lambda_{x_f} = 0$. The final value of λ_γ can be obtained from Eq. (48).

$$\lambda_{\gamma_f} = \frac{V_f}{g} \frac{1}{n_f + 1} \quad (62)$$

The optimal thrust parameter η^* is given in Eq. (41), depending on the signλ_E. As $\lambda_{E_f} = 0$, the value of η^* has to be determined by

$$\operatorname{sign}[\lambda_E(t_f - \Delta t)]_{\Delta t \to 0} = -\operatorname{sign}\dot{\lambda}_E(t_f) \quad (63)$$

If $q_f > q_c$, which can be the usual case for a split-S,

$$\dot{\lambda}_E(t_f) = (g/V^2) > 0 \quad (64)$$

clearly indicating by applying Eq. (55) that such a maneuver has to be flown, at least along the structural limit subarc, with zero thrust.

For $q_f < q_c$ the final value of $\dot{\lambda}_E$ is

$$\dot{\lambda}_E(t_f) = -\frac{g}{V^2} \frac{n_L^* - 1}{n_L^* + 1} < 0 \quad (65)$$

which leads us to conclude, as a consequence of Eq. (58), that along the entire lift-limited subarc, full thrust is optimal.

It can be summarized that a split-S initiated with $q_0 > q_c$ and a half-loop with $q_0 < q_c$ are flown with constant throttle setting. For other cases, an iterative solution is required to calculate the optimal thrust values.

In comparing aircraft turning performance in the vertical plane, a standard (probably nonoptimal) strategy is very often used. "Hard turn" pull-up maneuvers are computed with full thrust, while an optimal hard split-S is assumed to be flown with closed throttle and even with speed brakes extended. Such approximation is, however, very useful. Trajectory computations are straightforward and the results are representative. A full-thrust "hard" half-loop is probably near to optimal and undoubtedly a practical maneuver strategy in actual air combat.

In Fig. 4, typical half-loop trajectories of a hypothetical airplane flown at maximum thrust and load factor are plotted in the flight envelope. The double dotted line in this figure separates the flight envelope into two regions. From initial conditions above (or left) this line, no half-loop flown with the given strategy can be completed inside the flight envelope (i.e., with load factor $n_f^* \geq 1$).

It can be clearly seen that the air combat arena of such maneuvers is rather limited. In the example of Fig. 4, the line of $n_f^* = 1$ is below the line of the critical dynamic pressure q_c, indicating that almost all complete hard-turn maneuvers should begin with $n^* = n_{\max}$. Lines of equal maneuver time t_f or equal specific-energy loss $(\Delta E = E_0 - E_f)$ can also be plotted as a function of initial conditions and serve for performance comparison (see Fig. 5).

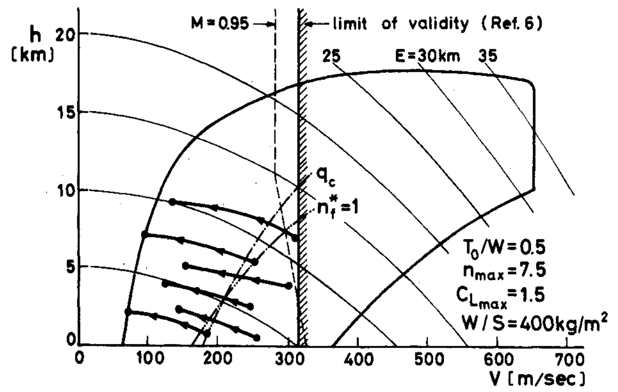

Fig. 4 Flight domain of hard pull-up maneuvers.

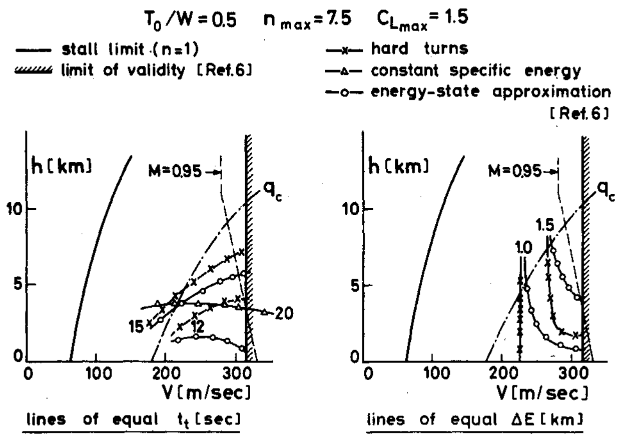

Fig. 5 Maneuver time and specific-energy loss mapping.

Energy-Conservative Vertical Turns

To avoid loss of specific energy during the maneuver, drag has to be kept equal to thrust. Such strategy is known as the "best sustained turn" in horizon maneuvering, and can be used equally in vertical turns.

Along a time-optimal conservative half-loop, three different subarcs may exist according to the pertinent factor limiting the instantaneous turning rate:

Structural limit load factor (thrust is less than maximum):

$$n^* = n_{\max}$$

$$\eta^* = \frac{D_0 + n_{\max}^2 D_i}{T_{\max}} \tag{66}$$

Maximum available thrust equals drag:

$$n^* = \left(\frac{T_{\max} - D_0}{D_i}\right)^{1/2}$$

$$\eta^* = 1 \tag{67}$$

Aerodynamic-lift limit (thrust is less than maximum):

$$n^* = n_L$$

$$\eta^* = \frac{D_0 + n_L^2 D_i}{T_{\max}} \tag{68}$$

This strategy can be easily implemented in an airplane requiring only a specific-energy-rate meter, introduced already in modern fighters. Using the maneuver strategy as defined by Eqs. (66-68), trajectories can be computed and used for performance evaluation. The minimum time to perform a half-loop using energy-conserving maneuver strategy is equal to or longer than the time using the "maximum instantaneous" strategy. The difference depends mainly on the thrust limited portion of the trajectory.

Drag-Efficient Vertical Turning

In the late sixties, a "most efficient" turning strategy was proposed by Col. (then Major) Boyd.[13] It defined the "optimal" load factor (or lift coefficient) which maximizes the ratio of the centripetal force and the aerodynamic drag. Such "optimal" control is independent of the aircraft-engine performance.

In vertical turning, the centripetal force is

$$N \stackrel{\Delta}{=} L - W\cos\gamma = W(n - \cos\gamma) \tag{69}$$

The ratio of this force to the aerodynamic drag can be written as

$$\frac{N}{D} = \frac{W(n-\cos\gamma)}{D_0 + n^2 D_i} \tag{70}$$

The load factor maximizing this ratio is

$$n^* = \cos\gamma + \left(\cos^2\gamma + \frac{D_0}{D_i}\right)^{1/2} \tag{71}$$

with

$$\frac{D_0}{D_i} = \left(\frac{q}{W/S}\right)^2 \frac{C_{D_0}}{K} \tag{72}$$

This "drag-efficient" strategy almost never makes use of the aircraft's best turning capability, and as a consequence, it requires a rather long time to complete a half-loop. Very often the maneuver is completed outside the dynamic flight envelope. From Eq. (71) it can be deduced that completion of such maneuvers with $n^* > 1$ requires that at the final state ($\gamma = \pi$) the dynamic pressure satisfies the following inequality:

$$q_f^2 \geq 3(W/S)^2 (K/C_{D_0}) \tag{73}$$

This strategy is probably useful only for the initial part of the pull-up maneuver ($\gamma \leq 60$ deg), where the requirement of $n^* > 1$ is always satisfied.

Suboptimal Feedback Solutions

The standard maneuvers discussed in the previous section are by no means optimal. They have been chosen for their simplicity, but are not representative for comparison of optimal aircraft performance. For this task, as well as for airborne applications, an approximate solution of the optical control problem, requiring minimal computational effort, can be an attractive candidate. In this section two types of such suboptimal solutions, expressed in a feedback form, are briefly summarized. Detailed description of the solution methodologies, being out of the scope of the present paper can be found in separate reports.[6,10]

Feedback Strategy for Vertical Turns Based on Energy-State Approximation

This recently developed "near-optimal" feedback strategy for vertical maneuvering is described in detail in Ref. 6. The energy-state approximation[14] is a well-known concept in flight mechanics. It assumes that the altitude, being a fast variable, can be considered as a control. For this assumption, optimality requires

$$\left.\frac{\partial H}{\partial h}\right|_E = 0 \tag{74}$$

yielding an algebraic relationship between the costates $(\lambda_\gamma, \lambda_E)$ and the other variables of the problem. Elimination of the control variables from Eqs. (43) and (74) leads to a feedback relationship of the form

$$(\lambda_\gamma / \lambda_E) = f(h, E, \gamma) \tag{75}$$

which determines, through Eq. (43), the optimal load factor as a function of the real state variables.

This solution is only suboptimal because the costate variables are not obtained by integrating Eqs. (32) and (33). It can be considered, however, as a continuous correction of the optimality conditions. The approximation compares very well with results obtained by exact solution of the original TPBVP and especially suits the weighted-sum payoff formulation. It represents a fine compromise between the hard-turn and energy-conservative maneuvers in the vertical plane, as can be seen in Figs. 5-7. This is an optimal compromise in the spirit of the "maximum maneuver principle"[13] imbedded in the payoff formulation. This solution satisfies the qualitative pilot requirement of "minimum time without excessive energy loss" stated in the Introduction. The domain of validity of this approximate solution is limited to subsonic speeds (see Figs. 4 and 5) and it cannot satisfy a prescribed final specific-energy requirement.

For such formulation a different type of solution is required.

Approximate Feedback Solution Based on Singular Perturbations

The concept of singular perturbation technique in problems of flight mechanics was introduced by Kelley[15,16] and applied since in different formulations by Calise[17,18] and Ardema[19] in the investigatons of optimal climb trajectories or nearly horizontal turns. This approximation technique can be used if there exists a physically small parameter ϵ in the problem, and if this parameter tends towards zero, the order of differential equation describing the dynamic system is reduced. Similar results can be achieved, however, if the variables of the system are separated by their time scales[15] as slow and fast parameters. The solution of the reduced-order model[16] is called the zero-order *outer* solution, or by analogy to fluid mechanics, the freestream. Such a solution (being of a reduced order) is unable to satisfy all the boundary conditons of the original problem. It has to be corrected by a so-called boundary layer or *inner* solution dominated by the fast variables.

In the boundary layer the slow variables are supposed to be known from the freestream. The *inner* solution has to satisfy

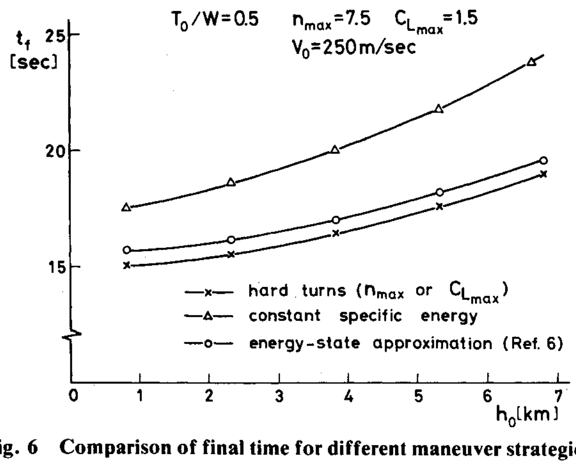

Fig. 6 Comparison of final time for different maneuver strategies.

the violated boundary conditions and match asymptotically the *outer* solution. It has been shown[18] that if the boundary layer is of the first order, a feedback solution is obtained and the asymptotic matching is easily satisfied.

A recent numerical investigation of optimal turning maneuvers[9] has shown that if a substantial energy loss is permitted by the prescribed value of E_f, the major part (including the terminal phase) of the maneuver is performed at an almost constant specific-energy level. A rapid loss of specific energy occurs at the initial phase of the pull-up. Based on this observation, it is proposed to select,[10] for the time-optimal problem defined in this paper, a half-loop maneuver flown at the prescribed final specific energy E_f, as a reduced-order solution.

In this freestream, the optimal controls $(\eta^*)^o$ and $(n^*)^o$ can be obtained directly from the constraints, as given by Eqs. (66-68). Using these controls a trajectory can be computed, along which the costate variables are obtained by backward integration and stored as a function of γ. The end conditions for such integration are given by Eqs. (29) and (48), yielding

$$\lambda_h^o(t_f) = 0$$

$$\lambda_\gamma^o(t_f) = \frac{V_f^o}{g}\,\frac{1}{(n_f^*)^o + 1} \tag{76}$$

In this way the complete solution of the suboptimal freestream of three state variables is obtained by a direct method without solving a TPBVP. This solution satisfies the terminal constraints $(\gamma_f = \pi,\ E_f)$ but not the initial condition $E_0 \neq E_f$. This condition has to be satisfied by the inner (boundary-layer) solution.

In the boundary layer, the time scale has to be stretched by the transformation

$$\tau = t/\epsilon \tag{77}$$

and, assuming that ϵ tends toward zero, only the differential equations for E^i and λ_E^i remain. The other variables will be taken equal to their values in the outer solution. This system of equations has to be solved with the initial condition $E(\tau=0) = E_0$ maximizing the Hamiltonian

$$H^i = -1 + \lambda_E^i\left(\eta\,\frac{T_{\max} - D_0 - n^2 D_i}{W}\right)V + \lambda_\gamma^o\,\frac{g}{V}\,(n - \cos\gamma)$$

$$+\lambda_h^o V\sin\gamma + \text{constraints} \tag{78}$$

As λ_γ^o and λ_h^o are known as functions of γ, the optimal control functions can be determined by λ_E^i and the state variables. Substituting λ_E^i from Eq. (43) into the Hamiltonian and equating to zero [see Eq. (48)] yields a quadratic equation for the optimal load factor

$$(n^*)^{i^2} - 2(n^*)^i\left[\cos\gamma + \frac{V}{g\lambda_\gamma^o}\,(1 - \lambda_h^o V\sin\gamma)\right] + \frac{T - D_0}{D_i} = 0 \tag{79}$$

It can be shown that this quadratic has two positive real roots. The larger root corresponds to negative specific-energy rate and it is the relevant one to the problem to be solved $(E_0 > E_f)$. When this solution does not violate the constraints λ_E^i will be positive and, as a consequence, full thrust $(\eta^{*i} = 1)$ has to be used. For constrained load factors the value of n_m^* or n_L^* has to be substituted into Eq. (78) to determine λ_E^i. Equation (79) is a feedback relation since λ_γ^o and λ_h^o are assumed to be known from the suboptimal outer solution as functions of γ. It can be used to integrate the original set of trajectory equations (1-4). The trajectory obtained tends towards the constant-energy freestream solution, satisfying both initial and prescribed terminal conditions for the specific

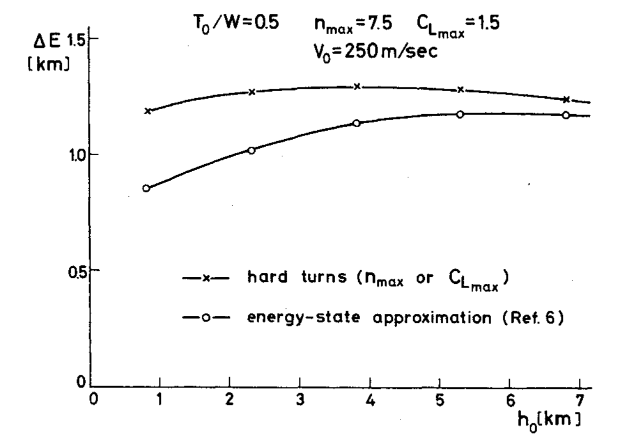

Fig. 7 Comparison of energy loss for different maneuver strategies.

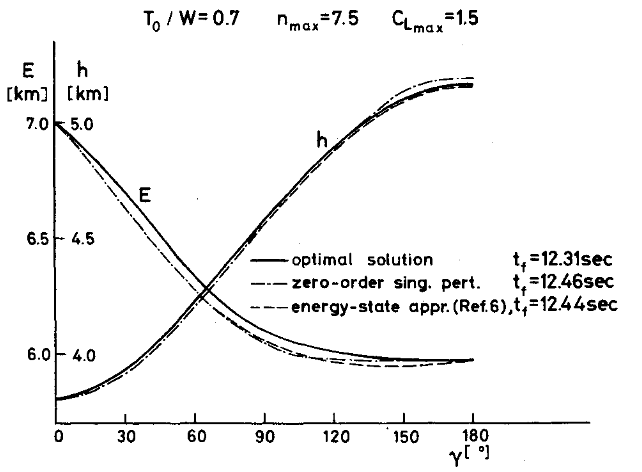

Fig. 8 Comparison of state-variable histories.

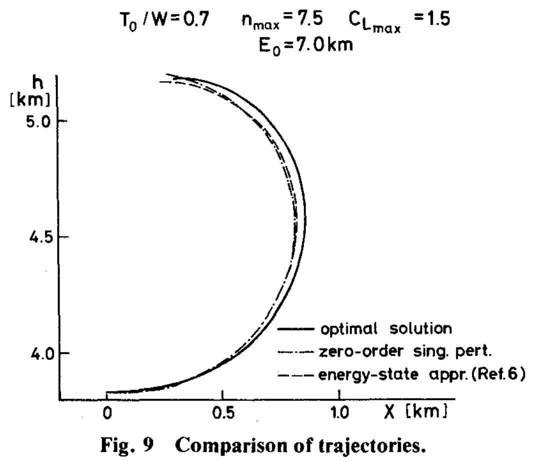

Fig. 9 Comparison of trajectories.

energy. This zero-order approximation gives fairly accurate results for a large set of parameters and end conditions.

Comparison to the Exact Optimal Solution

In Figs. 8 and 9 the state variables and the trajectories computed by both suboptimal methods are compared to the optimal solution of the original TPBVP. For the exact solution, a recent combined algorithm,[9] permitting the reduction of the numerical effort by more than 50%, was used. The example is of a hypothetical high-performance fighter design. The exact numerical solution of the TPBVP required 5.15 s of CPU on IBM 370/168. The suboptimal solutions, which provide a very fine approximation (better than 1.5%), were obtained with substantially less computational effort.

Conclusions

In the present paper, vertical turning performances of fighter aircraft are analyzed by methods which need only a minimum amount of computation. Based on genuine air combat requirements, an optimal control problem with two alternative payoff functions is formulated. Using a realistic mathematical model, the formal solution of this problem is expressed in terms of the costate variables. Further qualitative analysis dealing with sufficient conditions for optimality shows the nonoptimality of intermediate thrust, the admissible control combinations, and the rules of optimal thrust switching. The majority of these results are confirmed by the independent work of Uehara,[7,8] although he investigated a slightly different problem.

As the exact numerical solution of the optimal control problem is highly impractical for airborne applications (as well as for systematic performance assessment), two suboptimal methods of solution are outlined. These solutions require only a slightly higher numerical effort than the computaton of trajectories using standard maneuver strategies. They have the definite advantage of yielding a fine approximation of the optimal solution and, as a consequence, are much more representative for performance comparison of fighter aircraft. Moreover, in the suboptimal solutions the control variables are expressed in a feedback form, which is an attractive feature for real-time onboard implementations.

The topics discussed in this paper generate, for optimal maneuvering in the vertical plane, an insight which may eventually lead to more efficient air combat tactics and improved future aircraft design.

Appendix: Conditions for Singular Thrust Arc

The existence of an intermediate thrust ($0<\eta<1$) arc requires both $\lambda_E=\dot{\lambda}_E=0$. Substituting $\lambda_E=0$ into Eq. (32), multiplying by (V^2/g) and equating to zero leads to

$$\lambda_\gamma \frac{g}{V}(n-\cos\gamma)-\lambda_h V\sin\gamma-2\mu_L n_L V\frac{\partial n_L}{\partial V}=0 \qquad \text{(A1)}$$

Applying Eq. (48) for a singular thrust arc yields

$$\lambda_\gamma (g/V)(n-\cos\gamma)+\lambda_h V\sin\gamma-1=0 \qquad \text{(A2)}$$

From these two equations the costate variables λ_γ, λ_h on the singular arc can be expressed as the function of the state variables only. In view of Eq. (43), $\lambda_E=0$ dictates a limited load factor. The only other alternative, i.e., $\lambda_\gamma=\lambda_E=0$, can be directly discarded because it creates a contradiction between Eqs. (A1) and (A2). Therefore, the behavior of the singular arc and its optimality are examined separately on each of the constraint boundaries defined by Eqs. (19) and (20).

$|n^*|=n_{\max}<n_L(V,h)$ $(\mu_L=0)$

Linear combination of Eq. (A1) and (A2) yields for the costates

$$\lambda_\gamma=\frac{V}{2g(n_m^*-\cos\gamma)} \qquad \text{(A3)}$$

$$\lambda_h=\frac{1}{2V\sin\gamma} \qquad \text{(A4)}$$

As by Eq. (34) $\dot{\lambda}_h=0$ on the singular arc, Eq. (A4) indicates that this part of the trajectory has to be flown with

$$\dot{h}=V\sin\gamma=1/2\lambda_h=\text{const} \qquad \text{(A5)}$$

Time derivative of Eq. (A5) yields

$$\dot{V}\sin\gamma + V\cos\gamma\,\dot{\gamma} = 0 \tag{A6}$$

where $\dot{V}$ is obtained directly from the equations of motion combining Eqs. (1), (3), and (5):

$$\dot{V} = g\left(\frac{\eta T_{\max} - D}{W} - \sin\gamma\right) \tag{A7}$$

Substituting Eqs. (A7) and (2) into Eq. (A6) yields

$$(\eta T_{\max} - D)\sin\gamma = W(1 - n_m^*\cos\gamma) \tag{A8}$$

which is, of course, the force equilibrium condition in the vertical direction indicating $\ddot{h}=0$. From Eq. (A8) the singular value of the throttle parameter η can be derived and expressed by the feedback relation

$$\eta_s^*(V,h,\gamma) = \frac{D_0 + n_{\max}^2 D_i}{T_{\max}} + \frac{W}{T_{\max}}\,\frac{1}{\sin\gamma}\,(1 - n_m^*\cos\gamma) \tag{A9}$$

The same result can be obtained from $\ddot{\lambda}_E = 0$ after elaborate algebraic manipulations.

For optimality, the singular arc has to satisfy the generalized Legendre-Clebsch condition (Kelley-Contensou test),[20] which applies for the present case as

$$-\frac{\partial}{\partial\eta}\left[\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\eta}\right)\right] \leq 0 \tag{A10}$$

Differentiation of Eq. (36) for a singular arc ($\lambda_E = \dot{\lambda}_E = \mu_T = 0$) yields

$$\frac{\partial}{\partial\eta}\left[\frac{d^2}{dt^2}\left(\frac{\partial H}{\partial\eta}\right)\right] = \frac{VT_{\max}}{W}\,\frac{\partial\ddot{\lambda}_E}{\partial\eta} = -\frac{gT_{\max}}{V^2 W}\,\frac{\partial\dot{V}}{\partial\eta} \tag{A11}$$

Substituting Eq. (A7) indicates that the test fails and, as a consequence, a singular thrust subarc with $n^* = n_m^*$ is not optimal.

$|n^*| = n_L(V,h) \leq n_{\max}$

Combining Eqs. (A1) and (A2) leads, for this case ($\mu_L \neq 0$) and using Eqs. (47) and (20), to

$$\lambda_\gamma = -\frac{V}{2g\cos\gamma} \tag{A12}$$

$$\lambda_h = \frac{1}{2V\sin\gamma}\left(1 + \frac{n_L^*}{\cos\gamma}\right) \tag{A13}$$

The singular value of η^* is obtained from $\ddot{\lambda}_E = 0$,

$$\eta_s^*(V,h,\gamma) = \frac{D_0 + n_L^2 D_i}{T_{\max}} + \frac{W}{T_{\max}}\,\frac{1}{\sin\gamma}\left(1 + n_L^*\frac{\cos 2\gamma}{\cos\gamma}\right) \tag{A14}$$

Verification of optimality leads to an expression identical to Eq. (A11) indicating that the generalized Legendre-Clebsch conditions are also not satisfied for the case of $n^* = n_L^*$.

It can thus be summarized that the optimal vertical turning maneuvers, as formulated in this paper, cannot have an intermediate thrust ($0 < \eta < 1$) subarc. An exceptional case of an optimal intermediate thrust subarc may exist if a part of the maneuver is flown at the critical dynamic pressure q_c defined by Eq. (40). In this case, analyzed in Ref. 21, the available thrust has to satisfy the inequality

$$T_{\max} > D(q_c) + W\sin\gamma\left[1 + \frac{2q_c}{\rho^2 g}\left(-\frac{d\rho}{dh}\right)\right] \tag{A15}$$

requiring very high values, beyond the reach of present jet propulsion technology. Such intermediate thrust arc is, however, not a singular subarc in the classical sense. For its computation, a multiple-point boundary layer has to be solved.[21]

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